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An analytic family of $\mathcal{PT}$-symmetric Hamiltonians with real eigenvalues

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Abstract

We study nonlinear perturbative expansions for $\mathcal{PT}$-symmetric local Schrödinger operators. The Schrödinger operator is a sum of the harmonic oscillator Hamiltonian and a local $\mathcal{PT}$-symmetric potential depending, in general, nonlinearly on the perturbation parameter. A specific class of models having real spectrum for any value of the parameter is proposed.

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1. Introduction

In [1] the authors analyze the question of perturbation theory for $\mathcal{PT}$-symmetric Hamiltonians of the form $H(\alpha) = H_0 + \mathrm{i}\alpha W$, $\alpha \in \mathbf{R}$, where H_0 is a $\mathcal{PT}$-symmetric Schrödinger operator in $L^2(\mathbf{R})$ and $W \in L^\infty_{\mathrm{loc}}(\mathbf{R})$ is a real-valued function such that $\mathcal{P}W := W(-x) = -W(x)$; $\mathcal{T}$ is complex conjugation. More precisely, in [1] it is proved that, under suitable assumptions on H_0 and W, the perturbed eigenvalues $E(\alpha)$ of $H(\alpha)$, which converge to those of H_0 as $\alpha \to 0$, are real for $|\alpha|$ sufficiently small. The aim of this paper is to extend these results to Hamiltonians of the form

$$H(\alpha) = H_0 + \mathrm{i}W_\alpha \tag{1.1}$$

where the dependence of W_α on the perturbation parameter is no longer simply linear, but of the most general form. Operators of the form (1.1) often arise in problems of interest in physics insofar as relevant potentials are expressed in terms of exponential or trigonometric functions which, by construction, have to depend on dimensionless arguments. A very interesting result in this framework has been recently obtained in [2], where numerical calculations have shown the existence of non-real complex conjugate eigenvalues for suitable values of $\alpha > 0$ for the Hamiltonian

$$H(\alpha) = p^2 + x^2 + \mathrm{i}x^3\,\mathrm{e}^{-\alpha(p^2+x^2)}. \tag{1.2}$$

Since for $\alpha = 0$ one formally has $H(0) = p^2 + x^2 + \mathrm{i}x^3$ and for $\alpha = +\infty$ one has $H(\infty) = p^2 + x^2$, and the spectra of both $H(0)$ and $H(\infty)$ are purely real, one is led to conjecture that the perturbed eigenvalues of $H(\alpha)$ are real both for $\alpha \to 0$ and as $\alpha \to +\infty$, i.e., for α small and for α large. However, in (1.2) the perturbation term $\mathrm{i}W_\alpha = \mathrm{i}x^3\,\mathrm{e}^{-\alpha(p^2+x^2)}$ represents a $\mathcal{PT}$-symmetric pseudo-differential operator in $L^2(\mathbf{R})$ whose spectral analysis is not trivial; in fact, since it is not Hermitian there is no control *a priori* on its numerical range as $\alpha \to 0$. As a first approach to problem (1.1) in this paper we restrict ourselves to consider only local perturbations $W_\alpha, \alpha \geqslant 0$, i.e., functions depending only on x and not on $p := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$. The perturbations are thus multiplication operators by functions $W_\alpha(x)$, such as, for instance, $W_\alpha(x) = x^3\,\mathrm{e}^{-\alpha x^2}$, while $H_0 = p^2 + x^2$ is the Schrödinger operator associated with the one-dimensional harmonic oscillator. We will prove, under suitable assumptions, that the perturbed eigenvalues of $H(\alpha)$ are real for $\alpha > 0$ small, if the eigenvalues of the unperturbed Hamiltonian $H(0) := H_0 + \mathrm{i}W_0$ are real and simple. As in [1], the proof is based on a stability argument for the spectrum of $H(0)$ with respect to the family $\{H(\alpha) : \alpha > 0\}$, which guarantees that the multiplicity of the eigenvalues is preserved as $\alpha \to 0$. For the importance of the reality of the spectrum in $\mathcal{PT}$-symmetric quantum mechanics, see, e.g., [1, 3–18]. The paper is organized as follows. In section 2 we state and prove the results on the stability and reality of the perturbed eigenvalues. We briefly sketch a class of models where the spectrum is real for any value of the parameter. The construction of these models is based crucially on the nonlinear dependence on the parameter. Finally, our results are summarized in the conclusions.

2. Stability and reality of perturbed eigenvalues

Let, then, $H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2$ denote the selfadjoint operator in $L^2(\mathbf{R})$ associated with the harmonic oscillator, with domain $D(H_0) = D(p^2) \cap D(x^2)$, $p^2 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Moreover, let $W_\alpha, \alpha \geqslant 0$, denote a family of continuous, odd, real-valued functions: $W_\alpha \in C^0(\mathbf{R})$ and $W_\alpha(-x) = -W_\alpha(x), \forall\alpha \geqslant 0$, such that $W_\alpha(x)$ converges to $W_0(x)$ as $\alpha \to 0$, uniformly on the compact subsets of $\mathbf{R}$, i.e. for any compact set $K \subset \mathbf{R}$

$$\lim_{\alpha\to 0}\left(\sup_{x\in K}|W_\alpha(x) - W_0(x)|\right) = 0. \tag{2.1}$$

Then let $H(\alpha) = p^2 + x^2 + \mathrm{i}W_\alpha, \alpha \geqslant 0$, denote the closed operator in $L^2(\mathbf{R})$ with $C_0^\infty(\mathbf{R})$ as a core, defined by

$$H(\alpha)u = -u'' + x^2u + \mathrm{i}W_\alpha u, \qquad \forall u \in C_0^\infty(\mathbf{R}), \tag{2.2}$$

and let $D(H(\alpha))$ denote its domain. Clearly, $H(\alpha)$ is $\mathcal{PT}$-symmetric $\forall\alpha \geqslant 0$. Now let $\|v\|$ denote the L^2-norm of a function $v \in L^2(\mathbf{R})$ and let $(p^2 + x^2)^{1/2}$ be the selfadjoint positive square root of $p^2 + x^2$ as defined by the spectral theorem. Then for any $u \in C_0^\infty(\mathbf{R})$ we have

$$\begin{aligned}\|(p^2+x^2)^{1/2}u\|^2 &= \langle u, (p^2+x^2)u\rangle \leqslant \langle u, (p^2+x^2)u\rangle + |\langle u, W_\alpha u\rangle| \\ &\leqslant \sqrt{2}|\langle u, H(\alpha)u\rangle| \leqslant \sqrt{2}\|H(\alpha)u\|\|u\| \leqslant \frac{\sqrt{2}}{2}(\|H(\alpha)u\|^2 + \|u\|^2).\end{aligned} \tag{2.3}$$

This implies that $D(H(\alpha)) \subset D((p^2+x^2)^{1/2})$ and that $(p^2+x^2)^{1/2}$ is relatively bounded with respect to $H(\alpha)$. This in turn implies that $H(\alpha)$ has compact resolvents, and therefore a discrete spectrum, since $(p^2+x^2)^{1/2}$ has compact resolvents (see, e.g., [19]). Moreover, one can easily see that the numerical range $\mathcal{N}(\alpha) := \{\langle u, H(\alpha)u\rangle : u \in D(H(\alpha)), \|u\| = 1\}$ of $H(\alpha)$ is contained in the right half plane $\mathcal{R}_+ := \{z : \mathrm{Re}z \geqslant 0\}$. In fact

$$\mathrm{Re}\langle u, H(\alpha)u\rangle = \langle u, (p^2+x^2)u\rangle \geqslant 0, \qquad \forall u \in C_0^\infty(\mathbf{R}), \quad \forall\alpha \geqslant 0. \tag{2.4}$$

Hence $\sigma(H(\alpha)) \subset \mathcal{N}(\alpha) \subset \mathcal{R}_+$ and $\|(z - H(\alpha))^{-1}\| \leqslant |\mathrm{Re}\,z|^{-1}, \forall z \notin \mathcal{R}_+$. Here $\sigma(H(\alpha))$ denotes the spectrum of $H(\alpha)$. Let E_j, $j = 1, 2, \ldots$, denote the sequence of the (discrete) eigenvalues of $H(0)$. Finally, let us recall that an eigenvalue E of an operator is called simple (or non degenerate) if $m(E) = 1$, where $m(E)$ denotes the algebraic multiplicity of E. For a review of this notion in an analogous framework, see [1]. Here we only mention the fact that if an eigenvalue is simple, there is only one corresponding eigenvector and the associated nilpotent vanishes. In analogy with the result obtained in [1] for perturbation potentials W_α depending only linearly on α, we can now state the main result of this paper.

Theorem 2.1. *Under the above assumptions each eigenvalue E_j of $H(0)$ is stable with respect to the family $H(\alpha)$, $\alpha \geqslant 0$. In particular, if E_j is simple and real there exists $\alpha_j > 0$ such that for $0 < \alpha < \alpha_j$, $H(\alpha)$ has exactly one eigenvalue $E_j(\alpha)$ close to E_j:*

$$\lim_{\alpha\to 0} E_j(\alpha) = E_j \tag{2.5}$$

and $E_j(\alpha)$ is real. Conversely, if $E(\alpha) \in \sigma(H(\alpha))$ and $\lim_{\alpha\to 0} E(\alpha) = E$, then E is an eigenvalue of $H(0)$.

Remark 2.2.

(i) For the notion of stability for eigenvalues, see, e.g., [19, 20] and also the article [1] where, after recalling the definition, the authors illustrate the stability theory developed by Hunziker and Vock in [20]. This theory will be applied also here in order to prove the theorem. Remark that stability is a very powerful property, since it implies, roughly speaking, that the multiplicity of any unperturbed eigenvalue E is 'preserved' as the perturbation is switched on (i.e. when we pass from $\alpha = 0$ to $\alpha > 0$ suitably small); in other words, near E there are exactly $m = m(E)$ eigenvalues (counting multiplicity) $E(\alpha)$ of $H(\alpha)$, for α sufficiently small.

(ii) The statement on the reality of $E_j(\alpha)$ if E_j is simple and real, follows from the fact that the eigenvalues of a $\mathcal{PT}$-symmetric operator come in pairs of complex conjugate values. So $E_j(\alpha)$ has to be real, because otherwise there would be two distinct eigenvalues, $E_j(\alpha)$ and $\overline{E_j(\alpha)}$, near E_j and not just one as stated in the theorem.

According to [20], the proof of theorem 2.1 follows from the following

Lemma 2.3. *Let $H^*(\alpha)$ denote the adjoint operator of $H(\alpha)$, $\alpha \geqslant 0$. Then*

(1) For all $u \in C_0^\infty(\mathbf{R})$

$$\lim_{\alpha\to 0} H(\alpha)u = H(0)u \tag{2.6}$$

and

$$\lim_{\alpha\to 0} H^*(\alpha)u = H^*(0)u. \tag{2.7}$$

(2) There exist constants $a, b > 0$ and γ, $|\gamma| < \pi/2$, such that

$$\langle u, p^2 u\rangle \leqslant a\{\cos\gamma\langle u, H(\alpha)u\rangle + \sin\gamma\langle u, H(\alpha)u\rangle + b\langle u, u\rangle\} \tag{2.8}$$

for all $u \in C_0^\infty(\mathbf{R})$.

(3) For any $z \in \mathbf{C}$, there exist positive constants δ, n_0 and α_0 such that

$$d_n(z, H(\alpha)) \geqslant \delta > 0, \tag{2.9}$$

for all $n > n_0$ and $0 \leqslant \alpha < \alpha_0$, where $d_n(z, H(\alpha)) := \mathrm{dist}(z, \mathcal{N}_n(\alpha))$ and

$$\mathcal{N}_n(\alpha) := \{\langle u, H(\alpha)u\rangle : u \in D(H(\alpha)), \|u\| = 1, u(x) = 0 \text{ for } |x| < n\} \tag{2.10}$$

is the so-called 'numerical range at infinity' (see [1]).

Proof. Assertion (1) immediately follows from (2.1). Indeed, for $u \in C_0^\infty(\mathbf{R})$ let K be a compact set in $\mathbf{R}$ such that $u(x) = 0$ for $x \notin K$. Since $H^*(\alpha)u = (p^2 + x^2 - \mathrm{i}W_\alpha)u$, we have

$$\|(H(\alpha) - H(0))u\|^2 = \|(H^*(\alpha) - H^*(0))u\|^2 = \|(W_\alpha - W_0)u\|^2$$
$$= \int_K |W_\alpha(x) - W_0(x)|^2 |u(x)|^2 \,\mathrm{d}x \leqslant (\sup_{x\in K} |W_\alpha(x) - W_0(x)|)^2 \|u\|^2, \qquad (2.11)$$

and (2.11) goes to zero as $\alpha \to 0$ by (2.1). As for (2), we have

$$\mathrm{Re}\langle u, H(\alpha)u\rangle = \langle u, (p^2 + x^2)u\rangle \geqslant \langle u, p^2 u\rangle, \qquad \forall u \in C_0^\infty(\mathbf{R}), \qquad \forall \alpha > 0. \qquad (2.12)$$

Hence (2.8) holds with $a = 1$, any $b > 0$ and $\gamma = 0$. Finally we prove (3). If we set $D_n(\alpha) := \{u \in D(H(\alpha)) : \|u\| = 1, u(x) = 0 \text{ for } |x| < n\}$ we have

$$d_n(z, H(\alpha)) = \inf_{u \in D_n(\alpha)} |z - \langle u, H(\alpha)u\rangle|. \qquad (2.13)$$

Now, for $u \in D_n(\alpha)$ we obtain

$$|z - \langle u, H(\alpha)u\rangle| \geqslant |\langle u, H(\alpha)u\rangle| - |z| \geqslant \mathrm{Re}\langle u, H(\alpha)u\rangle - |z|$$
$$= \langle u, (p^2 + x^2)u\rangle - |z| \geqslant \langle u, x^2 u\rangle - |z| \geqslant n^2 - |z|. \qquad (2.14)$$

Thus, it follows from (2.13) that

$$\lim_{\substack{n\to\infty \\ \alpha\to 0}} d_n(z, H(\alpha)) = +\infty, \qquad (2.15)$$

and this proves (3).

□

Example 2.4.

(1) The potential function W_α can be chosen as in section 1: $W_\alpha(x) = x^3\,\mathrm{e}^{-\alpha x^2}, \alpha \geqslant 0$. Indeed it is easy to verify that $W_\alpha(x)$ converges to $W_0(x) = x^3$ as $\alpha \to 0$ uniformly on any compact set $K \subset \mathbf{R}$. More precisely, let $|x| \leqslant M$ for all $x \in K$; then for $0 < \alpha < 1$ we have

$$\sup_{x\in K} \left| x^3 \left(\mathrm{e}^{-\alpha x^2} - 1\right)\right| \leqslant \sqrt{\alpha} M^3 \sum_{n=1}^\infty \frac{M^{2n}}{n!} \leqslant \sqrt{\alpha} M^3\,\mathrm{e}^{M^2}, \qquad (2.16)$$

and (2.16) tends to zero as $\alpha \to 0$. Since the eigenvalues of $H(0)$ are real and simple, so is each perturbed eigenvalue $E_j(\alpha)$ of $H(\alpha)$ for $0 < \alpha < \alpha_j$. Now setting $\alpha' = \alpha^{-1}$ we can draw similar conclusions for $\alpha' \to 0$, i.e. $\alpha \to +\infty$. In fact $W_\alpha(x) = x^3\,\mathrm{e}^{-\alpha x^2}$ converges uniformly to $W_\infty(x) = 0$, uniformly in $\mathbf{R}$ as $\alpha \to +\infty$ and the eigenvalues of $H(\infty) := p^2 + x^2$ are real and simple.

(2) The functions $W_\alpha(x) = x^3 \cos(\alpha x)$, $W_\alpha(x) = x^3\,\mathrm{e}^{\alpha x^2}$ and $W_\alpha(x) = x^{2k}\sin(\alpha x), k = 0, 1, \ldots, \alpha \geqslant 0$, all satisfy the assumptions of theorem 2.1. Note that the function W_α is not required to be bounded.

Remark 2.5.

(a) Theorem 2.1 implies that (non-real) complex eigenvalues of $H(\alpha)$ cannot accumulate at finite points but only at infinity as $\alpha \to 0$. If, in general, our result does not exclude the possibility of complex eigenvalues (diverging to infinity as $\alpha \to 0$), however for the potential $W_\alpha(x) = x^3\,\mathrm{e}^{-\alpha x^2}$, because of its uniform boundedness and convergence to zero as $\alpha' = \alpha^{-1} \to 0$, it is possible to repeat the argument used in [4, 18] in order to prove that for $\alpha' = \alpha^{-1}$ suitably small the whole spectrum of $H(\alpha)$ is purely real. Indeed, one can prove that if α_0 is chosen in such a way that $|W_\alpha(x)| < 1$ for $\alpha > \alpha_0$, then the power series expansion for the resolvent $(z - H(\alpha))^{-1}$ is convergent for $|\mathrm{Im}\, z| > 1$. Moreover, any eigenvalue of $H(\alpha)$ is trapped inside a square centered at some eigenvalue E_j of $H_\infty = p^2 + x^2$ with side 2, and an analyticity argument in the parameter α shows that it coincides with the (real) perturbed eigenvalue $E_j(\alpha)$ whose existence (and reality) is guaranteed by theorem 2.1. In other words, in this particular case the set of perturbed eigenvalues $E_j(\alpha)$, $j = 1, 2, \ldots$, covers the whole spectrum of $H(\alpha)$, for $\alpha > \alpha_0$.

(b) The results obtained above and in [3, 4, 18] with perturbation theory methods ensure the reality of the spectrum for α small. Now we briefly argue (see [21] for more details) that the nonlinearity allows us to build a class of models where the reality of the spectrum holds independently of the values of the parameter. More precisely, the result is a consequence of the two requirements

$$\|W_\alpha\| \to 0 \quad \text{as } \alpha \to 0^+ \qquad \text{and} \qquad \|W_\alpha\| < \tfrac{1}{2}, \qquad \forall \alpha > 0. \tag{2.17}$$

A typical W_α is given by $W_\alpha(x) = c_k(\alpha) x^{2k+1}\,\mathrm{e}^{-\alpha x^2}$, $k = 0, 1, \ldots, \alpha > 0$, for a suitable choice of $c_k(\alpha)$. It is obvious that the second requirement in (2.17) cannot be satisfied if W_α depends linearly on α. Having assessed that the spectrum is purely real one could conjecture that for any value of $\alpha > 0$ a similarity transformation, depending on α, connects $H(\alpha)$ with a selfadjoint operator $h(\alpha)$. We remark that similarity transformations are currently discussed to link $\mathcal{PT}$-symmetric Hamiltonians to Hermitian Hamiltonians under suitable assumptions (see, e.g., [22]; see also [23, 24]).

3. Conclusions

A further class of $\mathcal{PT}$-symmetric operators admitting real eigenvalues is isolated. With respect to former results, the present class exhibits the peculiarity of a general dependence on the perturbation parameter, instead of the linear one always considered so far. It is precisely this nonlinear dependence which allows us to formulate a class of $\mathcal{PT}$-symmetric Hamiltonians with real spectrum independently of α as discussed in remark 2.5(b).

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